

Space-time dynamics from algebra representations *

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Abstract

We present a model for introducing dynamics into a space-time geometry. This space-time structure is constructed from a C^* -algebra defined in terms of the generators of an irreducible unitary representation of a finite-dimensional Lie algebra \mathcal{G} . This algebra is included as a subalgebra in a bigger algebra \mathcal{F} , the generators of which mix the representations of \mathcal{G} in a way that relates different space-times and creates the dynamics. This construction can be considered eventually as a model for $2 - D$ quantum gravity.

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A C^* -algebra constructed by means of the generators of a finite-dimensional Lie algebra \mathcal{G} is employed to construct an appropriate space-time geometry structure from strictly symmetry grounds. Moreover, a notion of dynamics (in a sense to be specified below) can be introduced into the scheme by considering \mathcal{G} as inserted in a bigger algebra \mathcal{F} which encloses the physical content of the model. The method somewhat parallels that of Madore in constructing the *fuzzy sphere* [1]. While he employed the generators in irreducible unitary representations of $su(2)$, we make use here of the algebra of $sl(2, \mathbb{R})$ to create what might be called *fuzzy hyperboloids* (there are different ways of introducing non-commutativity in hyperboloids, thus leading to different “fuzzy hyperboloids”; we shall give a precise meaning to *ours* below) .

For the sake of clarity and to avoid redundancies, let us postpone the detailed analysis of the concrete C^* -algebra of interest until we have chosen a specific model in which our statements acquire a completely defined meaning. For the moment let us accept that a geometry notion (an hyperboloid) can be related to each particular irreducible representation of \mathcal{G} .

To introduce dynamics into this context, we consider a bigger algebra \mathcal{F} with a central extension structure. This algebra contains $sl(2, \mathbb{R})$ as a subalgebra singularized by algebra (pseudo-)cohomology criteria: $sl(2, \mathbb{R})$ is in the kernel of the cocycle. Generators which give a central term in their commutators are dynamical and form conjugated pairs, while those in the kernel of the cocycle are the kinematical ones (this can be explicitly shown through the construction of a symplectic form). Then, we construct a unitary, irreducible representation

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of \mathcal{F} . Assuming complete reducibility of the representation under $sl(2, \mathbb{R})$, we find a collection of $sl(2, \mathbb{R})$ irreducible, unitary representations, each defining a space-time geometry, via the $sl(2, \mathbb{R})$ generators. The action of the rest of the generators in \mathcal{F} mix the different $sl(2, \mathbb{R})$ representations, defining dynamics in the ensemble of the *hyperboloids*.

Let us make the foregoing considerations more specific. For 2D quantum gravity motivations [2, 3, 4], we choose \mathcal{F} to be the Virasoro algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n,-m}, \quad (1)$$

where c is the true extension parameter and c' is a redefinition of L_0 (pseudo-cohomology), which must be taken into account in order to fully explore the dynamical content of the algebra [5] (the standard expression in the literature for (1) uses c and h instead of c and c' , where $h = \frac{c-c'}{24}$; parameters (c, c') are better suited for our discussion).

Then, a highest-weight representation of the Virasoro algebra, $\mathcal{H}_{(c,c')}$, is constructed. Unitarity, irreducibility [6, 7, 8, 9, 10] and the presence of $sl(2, \mathbb{R})$ in the kernel of the cocycle (i.e., $sl(2; R)$ is kinematical) are automatically guaranteed with the imposition of $c = c'$ and $c > 1$. This $sl(2, \mathbb{R})$ is generated by $\langle L_1, L_0, L_{-1} \rangle$.

The representation $\mathcal{H}_{(c,c)}$ is accomplished by imposing $L_n | 0 \rangle = 0$ (for $n \geq -1$, annihilation operators), and the states have the form:

$$L_{n_1} \dots L_{n_j} | 0 \rangle \quad (n_1, \dots, n_j \leq 2 \text{ creation operators}) \quad (2)$$

To consider the reduction of the representation under the kinematical subalgebra $sl(2, \mathbb{R})$, we look for $| N, 0 \rangle$ states, satisfying $L_1 | N, 0 \rangle = 0$, thus being highest-weight vectors for $sl(2, \mathbb{R})$. It can be shown [11] that each of these vectors must belong to a definite Virasoro level (i.e., they must be L_0 -eigenvectors). Using this fact, and denoting as $D^{(N)}$ the dimension of the Virasoro N level, we find $(D^{(N)} - D^{(N-1)})$ highest-weight vectors in the Virasoro N level (we need only notice that the operator L_1 restricted to $Level(N)$ with values in $Level(N-1)$ is an epimorphism, and $\dim(Level N) = \dim(Ker L_1) + \dim(Im L_1)$). An $sl(2, \mathbb{R})$ representation $R^{(N)}$, of Casimir value $N(N-1)$, is reached by the successive action of L_{-1} on each of the previously found $| N, 0 \rangle$ vectors: $| N, n \rangle = (L_{-1})^n | N, 0 \rangle$. Furthermore, the different $sl(2, \mathbb{R})$ representations are orthogonal (with the scalar product induced from the Virasoro algebra). The representation of the Virasoro algebra is thus completely reduced:

$$\mathcal{H}_{(c,c)} = \bigoplus (D^{(N)} - D^{(N-1)}) R^{(N)} \quad (3)$$

Note that the representation $R^{(N)}$ is degenerated and its weight $(D^{(N)} - D^{(N-1)})$ increases with N .

As stated above, the space-time is reconstructed from a C^* -algebra, and in the search of it we follow the spirit of Madore [1] in the realization of the “fuzzy sphere”, but with a different objective which results in a different construction.

The aim in [1] was to construct a non-commutative geometry for the sphere in such a way that the classical geometry is recovered in a certain limit. This was achieved through the construction of a succession of C^* -algebras the limit of which is the algebra $\mathcal{C}(S^2)$ of complex-valued smooth functions on the sphere.

The explicit sphere was defined by:

$$g_{ab} x^a x^b = r^2 \quad (r \text{ being a fixed radius}) \quad (4)$$

g_{ab} being the Killing metric.

To implement the n -th element of the succession of C^* -algebras, he defined “coordinates” from the J_n^a generators of an irreducible representation of dimension n of $su(2)$:

$$x_n^a = k_n J_n^a, \text{ (where } k_n \text{ is a constant with appropriate dimensions).} \quad (5)$$

When polynomials were considered in these non-commutative coordinates of order up to $n - 1$, with the Casimir constraint (4), an algebra isomorphic to M_n ($n \times n$ matrices) resulted. This non-commutative C^* algebra M_n was then used to construct a matrix geometry which in the limit $n \rightarrow \infty$ goes to the standard geometry on the sphere of radius r .

Geometry becomes fuzzy in this process. For each matrix geometry M_n , points are replaced by states of the n -dimensional $su(2)$ representation considered. We can prove that $k \rightarrow 0$ in the limit $n \rightarrow \infty$, and thus coordinates become commutative, allowing the characterization of a point by the use of two coordinates (recovering the standard notion of a point).

In our case, the starting algebra is $sl(2, \mathbb{R})$ instead of $su(2)$, so that *hyperboloids* substitute spheres when the Casimir constraint is imposed.

C^* -algebras are again built from the representations of our Lie algebra (and thus we are in the spirit of Madore), but now we are not trying to approximate any previous classical geometry (true hyperboloids of “radii” r), and thus a succession of representations of the algebra for implementing such an approximation is not required. In our case, there is no arbitrariness in the $sl(2, \mathbb{R})$ representations we must consider. They are specific ones and are given by the reduction of the $\mathcal{H}_{(c,c)}$ representation. Each $sl(2, \mathbb{R})$ representation, $R_i^{(N)}$ (i for degenerate representations), will generate a (different) space-time geometry.

To construct the C^* -algebras, we first define the coordinate variables from the generators of $sl(2, \mathbb{R})$. Generators in the N -th $sl(2, \mathbb{R})$ representation are multiplied by an dimensional constant $k^{(N)}$ in order to get appropriate space-time coordinates:

$$\begin{aligned} x_i^{(N)} &= k^{(N)} L_i^{(N)} \quad i = -1, 0, 1 \\ \text{where } L_i^{(N)} &= L_i \big|_{N\text{-th } sl(2, \mathbb{R}) \text{ representation}} . \end{aligned} \quad (6)$$

We impose the condition that all the *hyperboloids* derived from the different $sl(2, \mathbb{R})$ representations in the Virasoro representation have the same *radius*, R , and this fixes the value of the constants $k^{(N)}$. The way the radius is implemented in an $sl(2, \mathbb{R})$ irreducible representation is, again, via the value of the Casimir on it (in fact, we have imposed irreducibility on these representations in order to have a well defined value of the Casimir):

$$-R^2 = g^{jk} x_j^{(N)} x_k^{(N)} = k^{(N)^2} N(N-1), \quad (7)$$

where g^{jk} is the $sl(2, \mathbb{R})$ Killing metric (note the condition $\frac{k^{(N)^2}}{|k^{(N)^2}|} = -1$, i.e., $k^{(N)}$ is a purely imaginary number).

Thus, we finally have:

$$\begin{aligned} R^2 &= -k^{(N)^2} N(N-1) \\ k^{(N)} &= i \frac{R}{\sqrt{N(N-1)}}. \end{aligned} \quad (8)$$

This is the way space-time variables are defined. To implement the C^* -algebra, we do not restrict ourselves to polynomials up to a certain order (we are not trying to define a sequence of space-times), but rather, we consider the entire enveloping algebra of these $x_i^{(N)}$, modulus the ideal generated by the Casimir (radius) constraint. Thus,

$$\begin{aligned} C^* - algebra &= Env(\langle x_{-1}^{(i)}, x_0^{(i)}, x_1^{(i)} \rangle) / Radius \\ Radius &= -g^{jk} x_j^{(N)} x_k^{(N)} = R^2. \end{aligned} \quad (9)$$

As can be seen from commutators among the space-time coordinates $[x_i^{(N)}, x_j^{(N)}] = k^{(N)} C_{ij}^k x_k^{(N)}$, this is a non-commutative C^* -algebra leading to a non-commutative geometry. Points are again replaced by states in the representation of $sl(2, \mathbb{R})$, and thus we have indeed *fuzzy hyperboloids*. For a better understanding of these “fuzzy” points, it is useful a glance at the indetermination relations, under which space-time is divided into cells:

$$\Delta x_i^{(N)} \Delta x_j^{(N)} \geq |k^{(N)}|^2 = \frac{R^2}{\sqrt{N(N-1)}}. \quad (10)$$

Different fuzzy hyperboloids of the same radius R are simultaneously found inside the Virasoro representation. These are distinguished by point density, which grows with the value of N , as can be seen from (10). We note that for large R values and very small N , the size of the cells is comparable to that of the hyperboloid. On the contrary, for a fixed R , we can find values of N as large as we wish, making $k^{(N)} \rightarrow 0$, so that cells tend to points and the space-time coordinates become commutative, *recovering* the classical geometry.

Our model for space-time is not just one of these hyperboloids, but the whole ensemble of them (they can be seen as different copies of the same hyperboloid, with equal R , but with different degrees of *fuzziness*, different N). We understand “point” to mean a normalized state in the Virasoro Hilbert space $\mathcal{H}_{(c,c)}$. Taking advantage of the complete reduction of $\mathcal{H}_{(c,c)}$ under $sl(2, \mathbb{R})$, this point can be written as a linear superposition of normalized vectors over the $sl(2, \mathbb{R})$ representations. Each of $sl(2, \mathbb{R})$ states is interpreted as a “point” in a concrete *fuzzy hyperboloid*, and thus the original point is spread over different hyperboloids. Indeed, it makes sense to consider the probability of the “point” to be in a concrete hyperboloid using the orthogonality of the $sl(2, \mathbb{R})$ representations and developing a standard quantum mechanical interpretation.

It is not our aim here to give a detailed analysis of the *fuzzy hyperboloid* geometry and its classical (large N) limit. We simply mention general features. The role of space-time diffeomorphisms is played by automorphisms of the C^* -algebra. Vector fields are derivations of this algebra; that is, linear mappings that satisfy the Leibnitz rule. These fields do not form a module over C^* , suggesting that we should consider one-forms as the fundamental objects having a bimodule structure over C^* [12]. From this, and the Killing metric on $sl(2, \mathbb{R})$, we could even define a metric and a connection (we do not enter into these details, which are subtle and deserve a specific study; basically, we aim to identify a proper C^* -algebra).

Let us focus now on the way the dynamical degrees of freedom enter the model. Since the motivation for the use of the Virasoro algebra is $2 - D$ gravity, we shall refer to these modes as “gravitational” ones. Their action on points (normalized states in the Virasoro Hilbert space) must be such that it preserves the norm (keeping the notion of “point”). Therefore, they must

be implemented by unitary transformations generated by hermitian operators. Starting from the condition

$$L_n^+ = L_{-n}, \quad (11)$$

hermitian combinations can be defined:

$$\begin{aligned} G_n &= L_n + L_{-n}, \quad n \geq 2 \\ G_{-n} &= i(L_n - L_{-n}), \quad n \geq 2, \end{aligned} \quad (12)$$

generating the unitary gravity transformations:

$$U_n = e^{i(k_n G_n + k_{-n} G_{-n})}. \quad (13)$$

Gravity transformations do not preserve the $sl(2, \mathbb{R})$ irreducible representations, so that if we start from a point which completely lies on a space-time of point density given by N , after the action of gravity this point is transformed into a superposition of points in different space-times of the same radius (the same Virasoro representation) but different N (different point density). This is the essence of the dynamics in the model: the Universe is not one of these space-times, but the whole ensemble of them, and a point is a superposition of states (eigen-points) spread over different-density space-times, the coefficients of which give the probability for the point to be in the corresponding space-time (because $sl(2, \mathbb{R})$ representations are orthogonal, a fact which allows the construction of proper orthogonal projectors), thereby defining a probability distribution of the point. The effect of (gravity) dynamics is that of changing this probability distribution (quantum motion of the point).

Space-times with different densities have different weights, in such a way that denser ones (more “classical” ones) are more abundant. Furthermore (as is easily checked given that we have a maximum weight representation), the repeated action of gravity generators move the density distribution toward larger N ; that is, gravity has a definite direction toward classical space-times. Combining this with the fact that classical space-times are the most abundant ones ($D^{(N)} - D^{(N-1)}$ increases with N), we could explain why Universe geometry is almost classical. If this construction is considered as a model for gravity, it must be remembered that one is not dealing with Einstein gravity, but rather with a higher-order correction to it (probably more related to a Wess-Zumino-Witten-like gravity). (Non-commutative) Einstein gravity should be studied in each of the hyperboloids that appear in the model, by introducing a metric connection notion with a dynamical content. In two dimensions, classical Einstein gravity is trivial, and thus we have not concerned ourselves with it. However, in higher dimensions this problem should be faced. We stress that the model is not incompatible with, but rather defines a framework to study, Einstein gravity.

As regards space-time operators, one must construct hermitian operators to give an observable character to the position of a point. Thus,

$$\begin{aligned} x_u^{(N)} &= x_1^{(N)} + x_{-1}^{(N)} \\ x_v^{(N)} &= i(x_1^{(N)} - x_{-1}^{(N)}). \end{aligned} \quad (14)$$

In this variables the $sl(2, \mathbb{R})$ Casimir constraint is given by:

$$R^2 = x_u^{N^2} + x_v^{N^2} - x_0^{N^2}. \quad (15)$$

This expression does not distinguish between *de Sitter* and *anti-de Sitter* space-times, which in two dimensions are topologically identical. In fact, the reconstruction of a geometry from a C^* -algebra does not provide a metric structure and thus such a distinction should not be expected at this level. There is freedom in choosing any of these by selecting an appropriate form of the $SL(2, \mathbb{R})$ Killing metric on the (x_u^N, x_v^N, x_0^N) space, which induces the corresponding metric on the hyperboloid through (15).

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